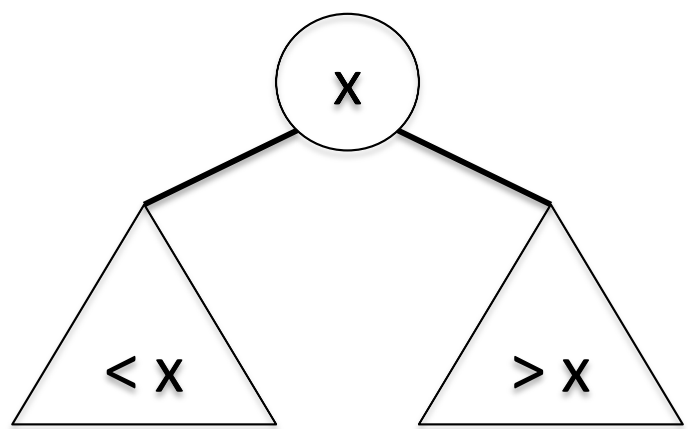
Balanced Binary Search Trees

9/12/24

Recall: Binary Search Trees (BSTs)

- Tree with key stored at each node such that
  – Every node has 2 children (left and right)
    - Children can be null (nodes have 0, 1, 2 actual children)
  – All keys in a node's right subtree are greater than its key and all in the left subtree are less



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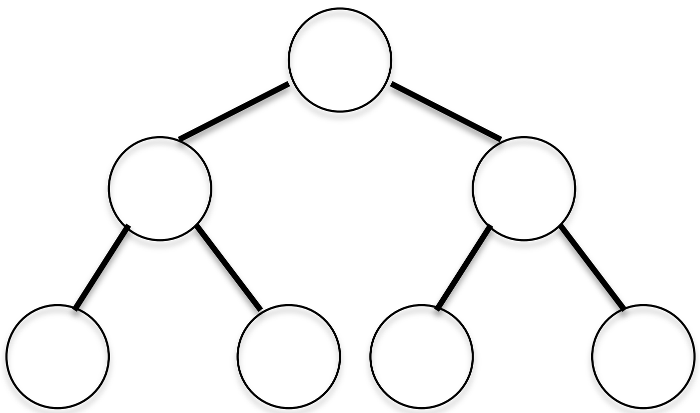

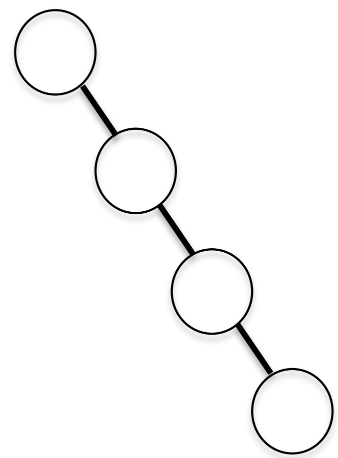

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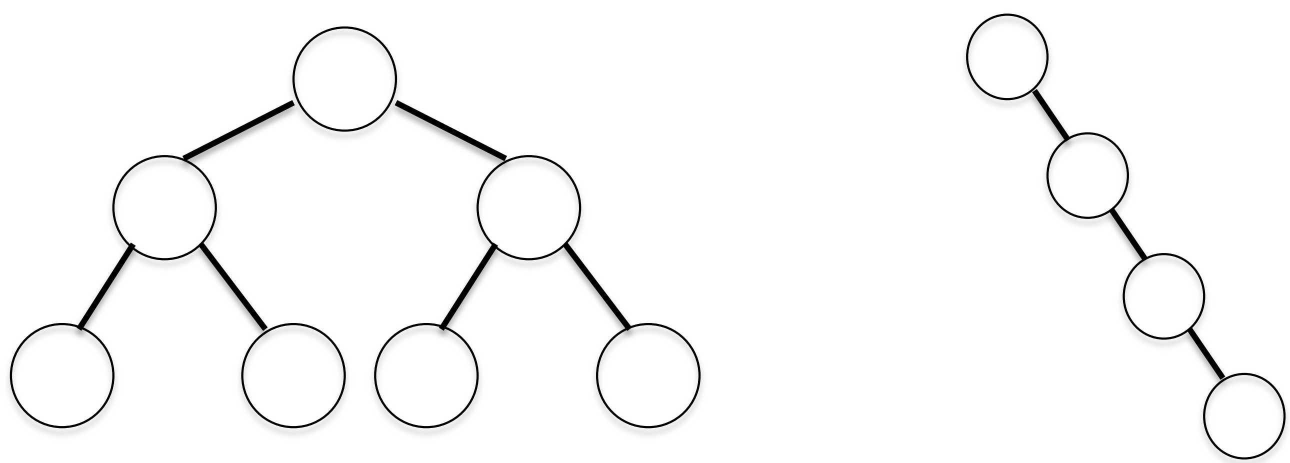

- A tree is *balanced* if its height is O(log n)

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C. 16
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E. None of the above

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Pf: $n <= 2^{h+1} - 1$

$n+1 <= 2^{h+1}$

$\log_2 (n+1) <= \log_2 2^{h+1} = h+1$

$h >= \log_2 (n+1) - 1 = \Omega(\log n)$

Induction framework

- Base case(s):
  - claim is true for one or more small values
- Induction hypothesis (IHOP)
  - assume claim is true up to some point
- Induction step
  - show that claim holds for next value (using IHOP)

Prove that $1+2+3+...+n = n(n+1)/2$

- Be sure to:
  - Clearly mark the IHOP and when it’s used
  - Work from one side (typically the left) and get the other side; don’t perform operations on both sides

Prove using induction: A full binary tree of height h has $2^{h+1}-1$ nodes

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Need strong induction: Assume claim holds for all smaller values

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$$F_n = \frac{1}{\sqrt{5}}\left(\left(\underbrace{\frac{1+\sqrt{5}}{2}}_{\phi}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

AVL trees

Height of an arbitrary node

- Height of a node is the height of the subtree rooted at that node

AVL tree

[Adelson-Velsky and Landis, 1962]

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- AVL tree: Every node has balance in {0,+1,-1}

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$$\phi$$

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$h = O(1.44\ldots \log_2 n)$

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