$F(0) = 1$

$F(i+1) = 2^{F(i)}$ "tower"

$\log^* N$ = least i s.t. $F(i) \ge N$

= least i s.t. $\underbrace{\lg \lg \lg \cdots \lg}_{i \text{ times}} N \le 1$

Lemma. There are $\le \frac{N}{2^r}$ nodes of rank r

Lemma: $u \to v$ implies u.rank < v.rank

$G(v)$ = Group(v) = $\log^*(v.\text{rank})$

Lemma: $G(v) \le \log^* N$ $\forall v$

Th: Total time for M operations on N items is $M \log^* N$

Pf: focus on finds

X_i = all nodes on a path during find

$X_i = Y_i \cup Z_i \cup W_i$

Y_i: root or child of root — ≤ 2

Z_i: parent change & $G(v) < G(v.p)$ — $\le \log^* N$

W_i: parent changes and $G(v) = G(v.p)$

$$\sum_{finds} |W_i|$$

$$= \sum_{nodes\ v} \#\text{finds s.t. } v \in W_i$$

rank(parent) increases each time

$$\leq \sum_{g=0}^{\log^* N} (\#\text{nodes in group } g) F(g)$$

#nodes for a particular group:

$$\sum_{r=F(g-1)+1}^{F(g)} \frac{N}{2^r} \leq \frac{N}{2^{F(g-1)+1}} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) = \frac{N}{F(g)}$$

$$\leq \sum_{g=0}^{\log^* N} \frac{N}{F(g)} F(g) = O(N \log^* N) = O(n \log^* n)$$